Geodesics Geometry

Joseph D. Clinton

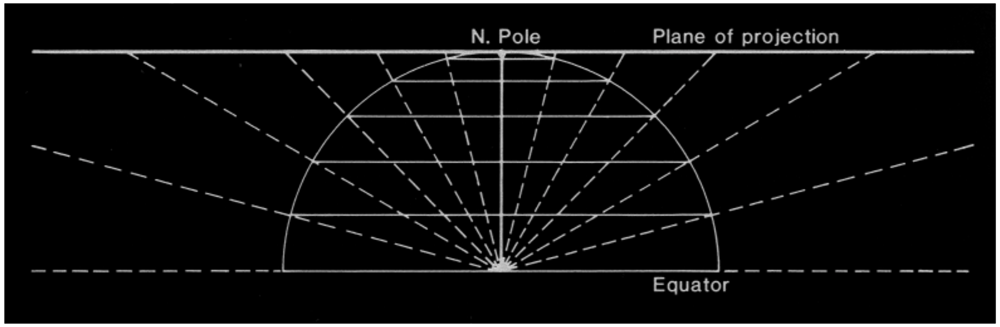
SNEC-04
June 2003

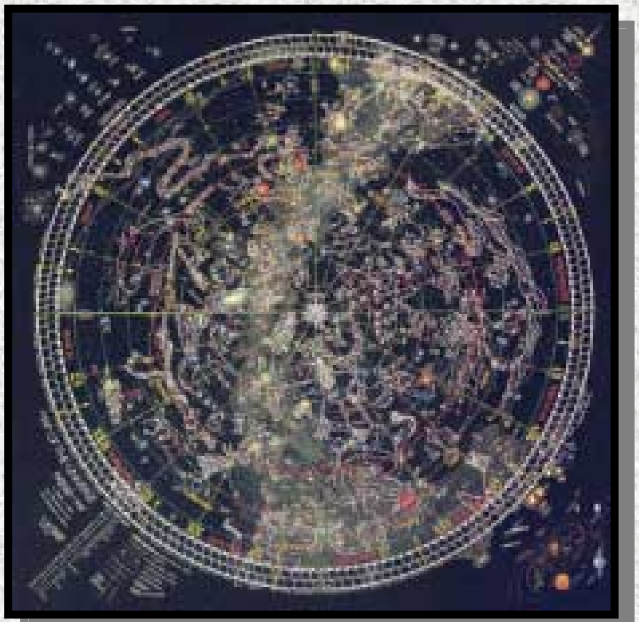
Introduction

- **Inverse Gnomonic Projection (Jena Planetarium)**
- **Fuller's Methods**
- **Kitrick's Algorithms**
- **Other Techniques**

Inverse Gnomonic Projection

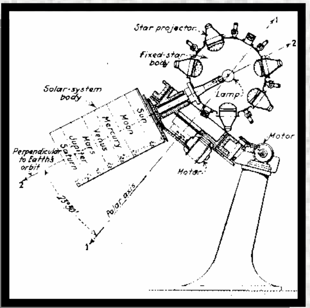
- The “horologium” (Gnomonic) projection was used in early times for constructing star maps of the heavens.



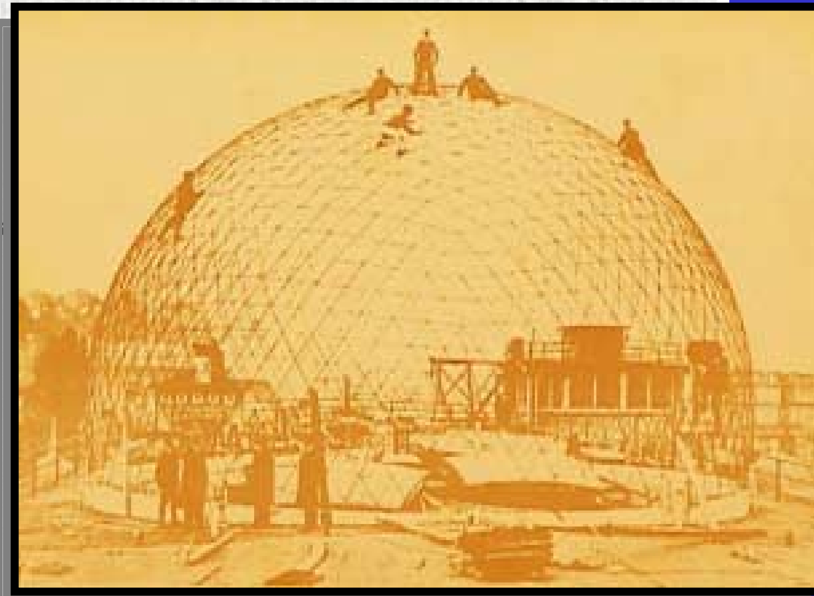


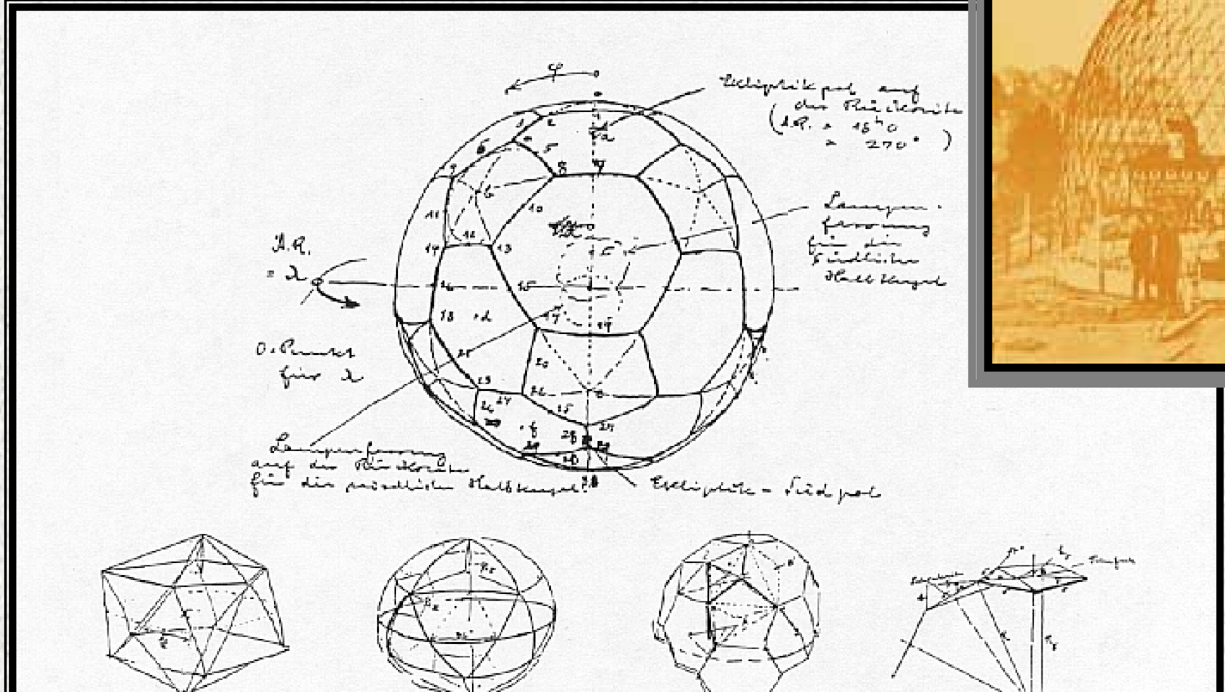
- Projection of a point on a spherical surface to a plane tangent to the sphere along a line passing through the point and the spheres center.

Jena Planetarium 1922

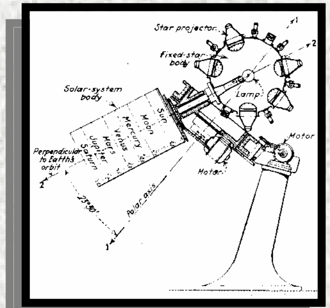


- An inverse gnomonic Projection was used

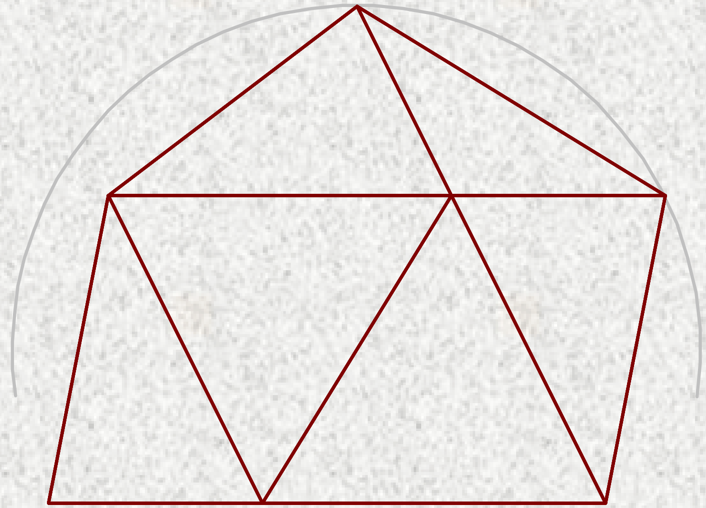


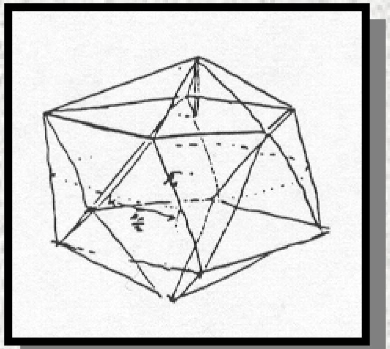


Walter Bauersfeld 1922



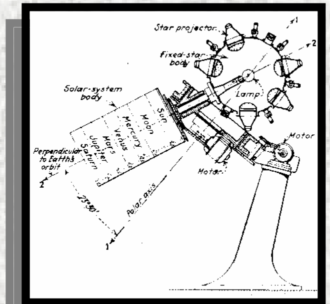
Jena Planetarium 1922



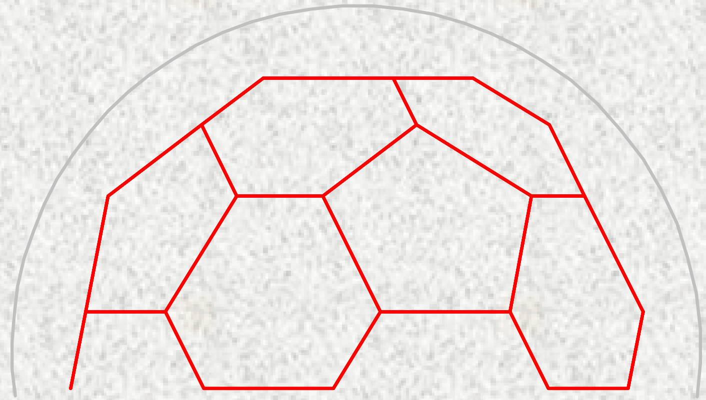


- "One starts with the fimiliar regular solid whose surface consists of 20 equilateral triangles..."

Walter Bauersfeld 1922

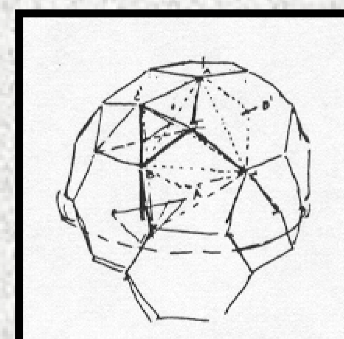


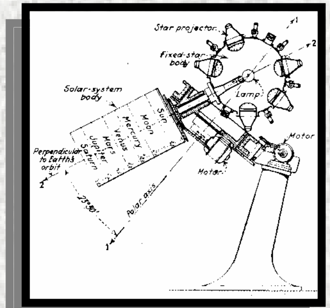
Jena Planetarium 1922



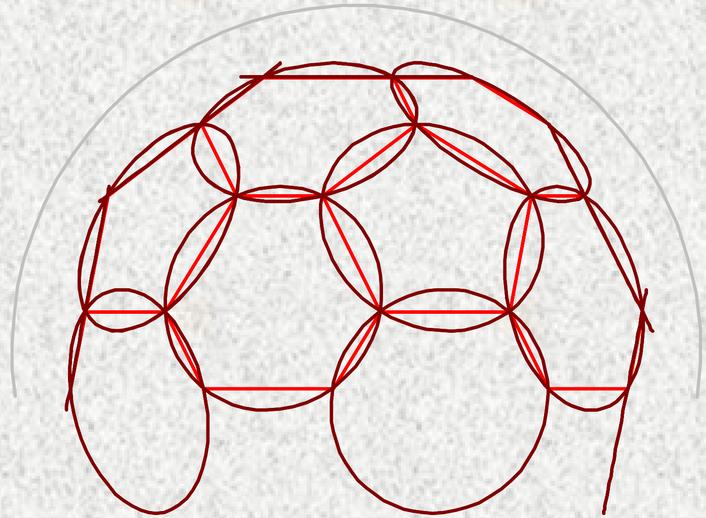
- "... and makes a straight cut across each of the 12 vertices which this solid possesses, then 20 hexagons and 12 pentagons are formed on the surface..."

Walter Bauersfeld 1922



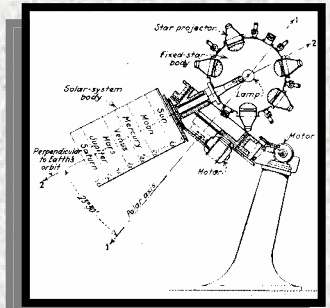


Jena Planetarium 1922

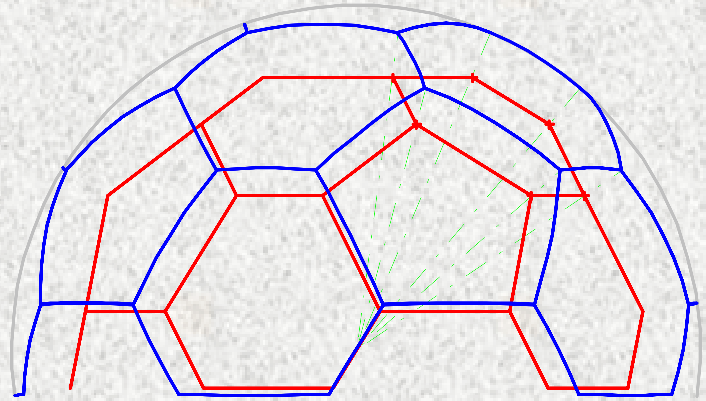


- "... With the cuts in the right places, it is easy to ensure that the circles circumscribing the pentagons and hexagons are equal..."

Walter Bauersfeld 1922

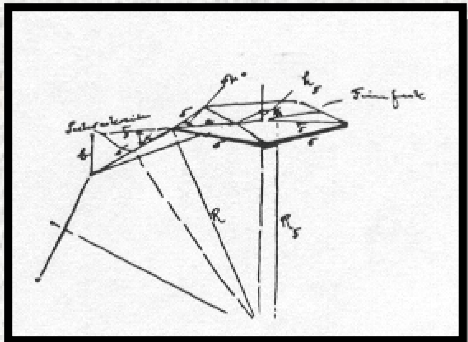


Jena Planetarium 1922

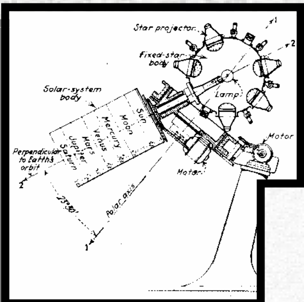


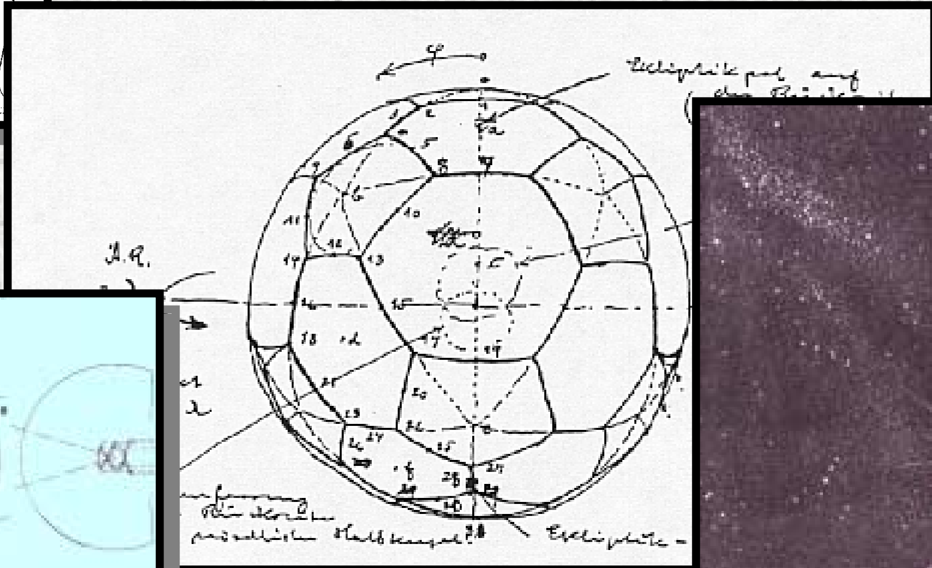
• "... If one imagines the edges of this solid projected out from the center onto a spherical surface with the same center, then the division of the sphere as described is formed."

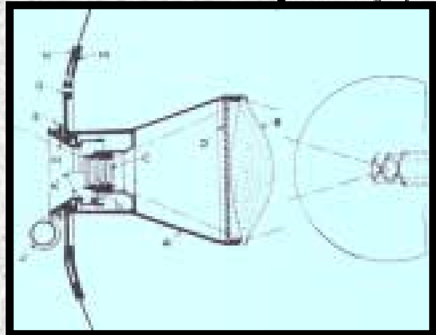
Walter Bauersfeld 1922

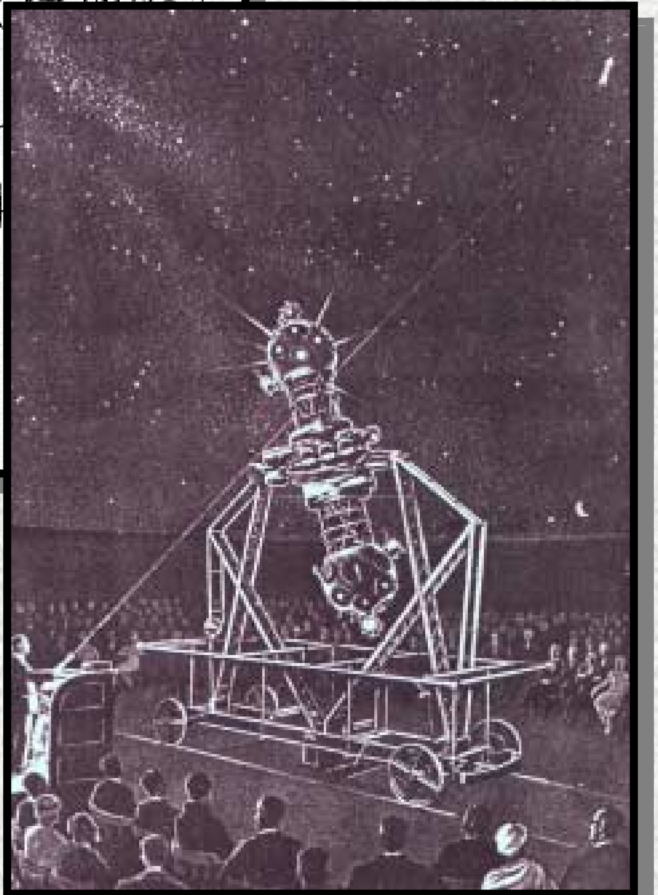


Jena Planetarium 1922



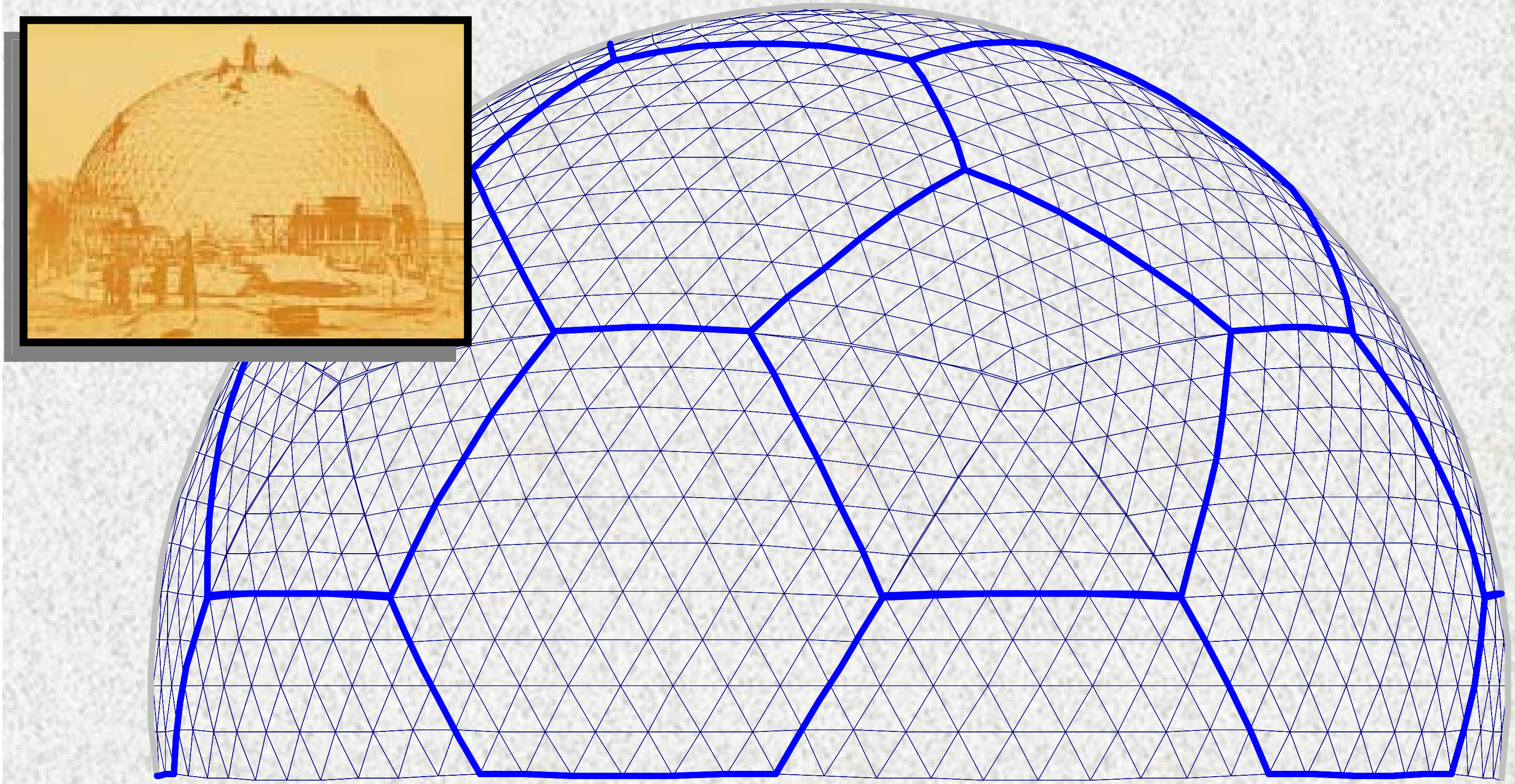






• **The lenses used were identical for each field of the truncated Icosahedron thus requiring the irregular form of the polyhedron.**

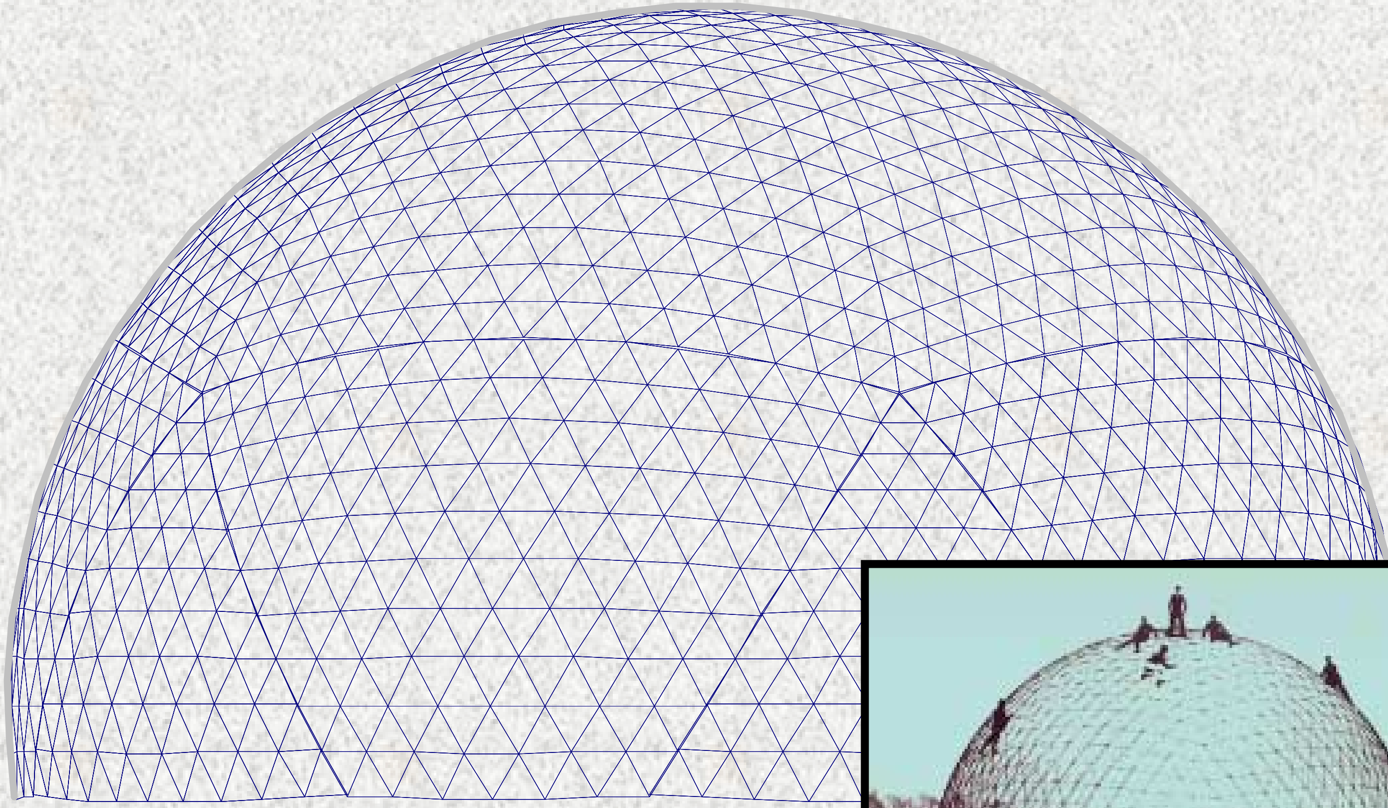
Jena Planetarium 1922



• Equal edge subdivision and triangulation of the hex/pent planes was done.

• The grid was transferred to the spherical surface using inverse gnomonic projection (Walter Bauersfeld – 1922)

Jena Planetarium 1922



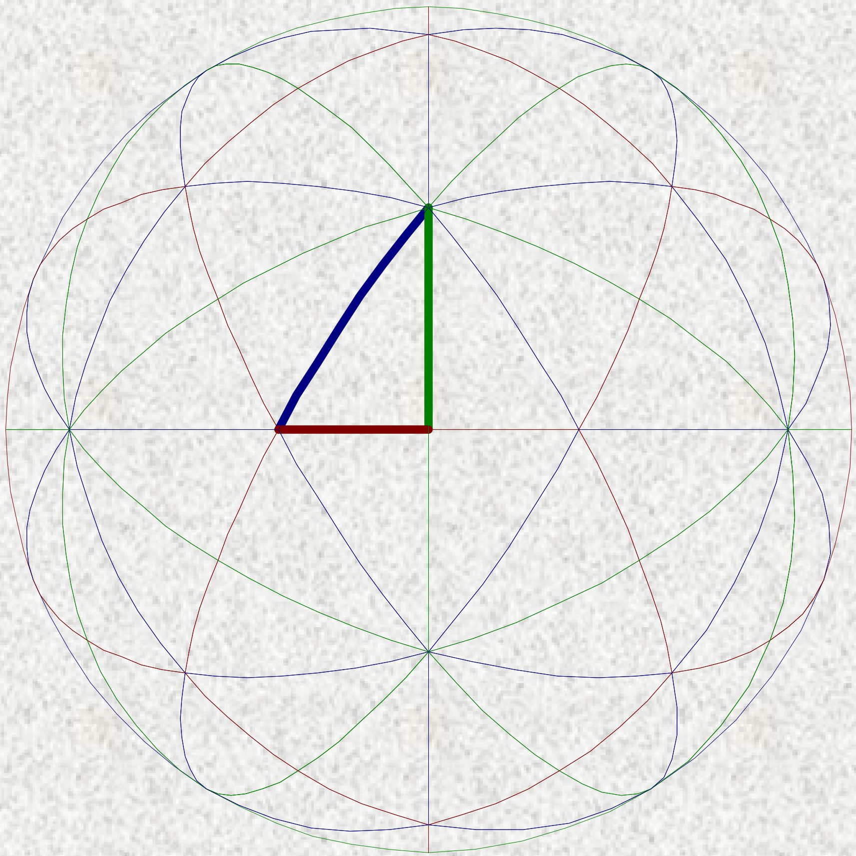
• A 16ν Class I Method 1 Geodesic Icosahedron (Clinton)

• Radial (Kitrick-1990)

Gnomonic Projection

- **The gnomonic projection method has become the most popular method among the general users interested in Geodesic domes.**
- **NASA documents Clinton - 1960's**
- ***Dome Books* popularized this method 1970-71**
- ***geodesic math* book Hugh Kenner 1976**
- ***finding chord factors of geodesic domes* Fred Blaisdell, Art Indelicato**
- ***Spherical Models* Magnus Wenninger**
- **GEODO software**
- **WinDome 48 software**
- **CADREY software**
- **GEODESIC software**
- **TekCAD software**
- **Forman software**

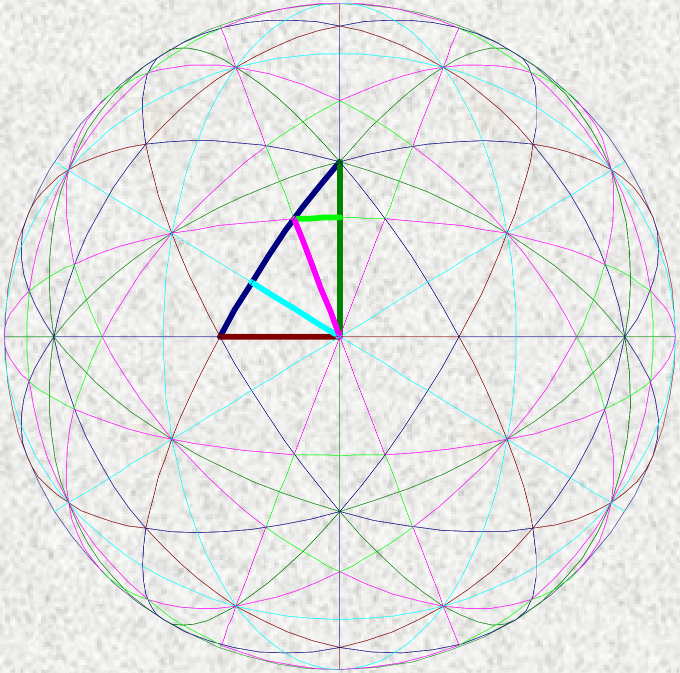
Fuller’s Methods



- “A problem proposed and solved by Schwarz in 1873: to find all spherical triangles which lead, by repeated reflection in their sides, to a set of congruent triangles covering the sphere a finite number of times.”

- There are only 44 kinds of Schwarz triangles.

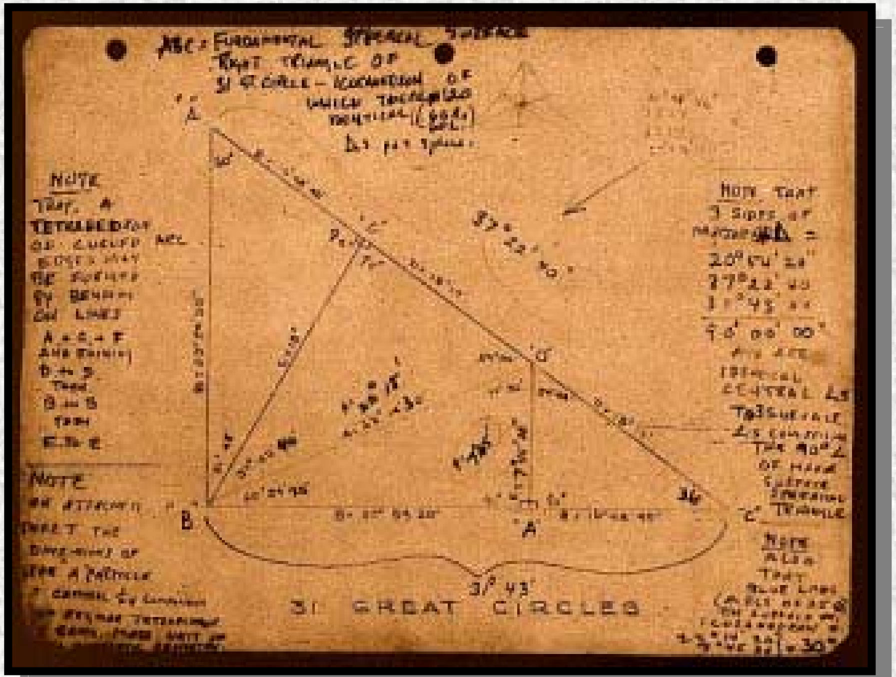
Fuller's Methods

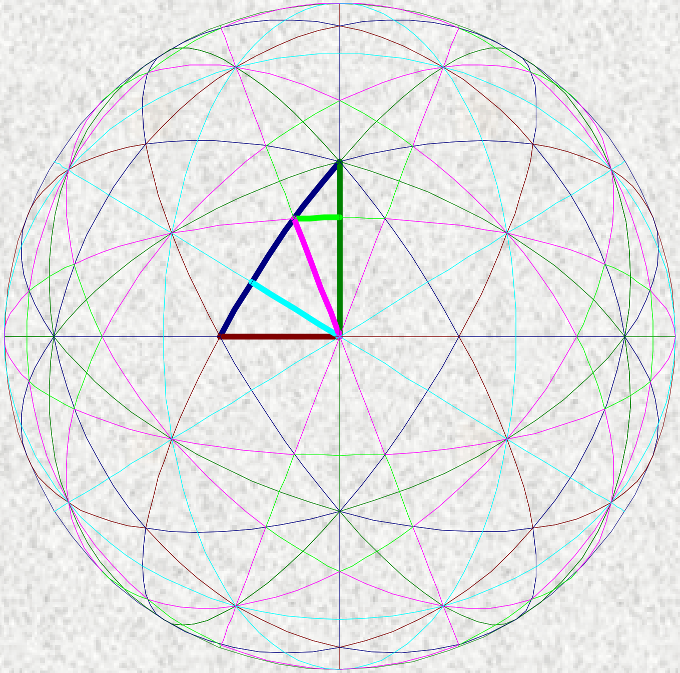


• Fuller's studies of the thirty-one great circles became the topological basis for his subdivision of the icosahedron into smaller cells that would describe his geodesic domes.

• The basic unit of the 31 great circle intersections is a Schwarz triangle of 1/120th of the sphere

Fuller’s Methods



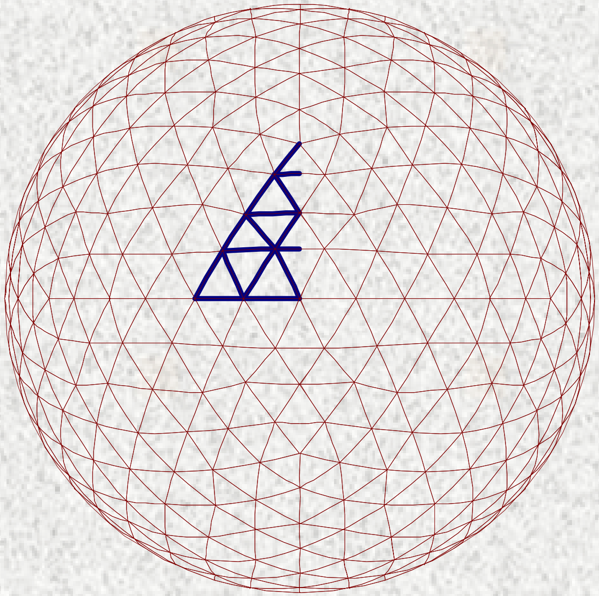


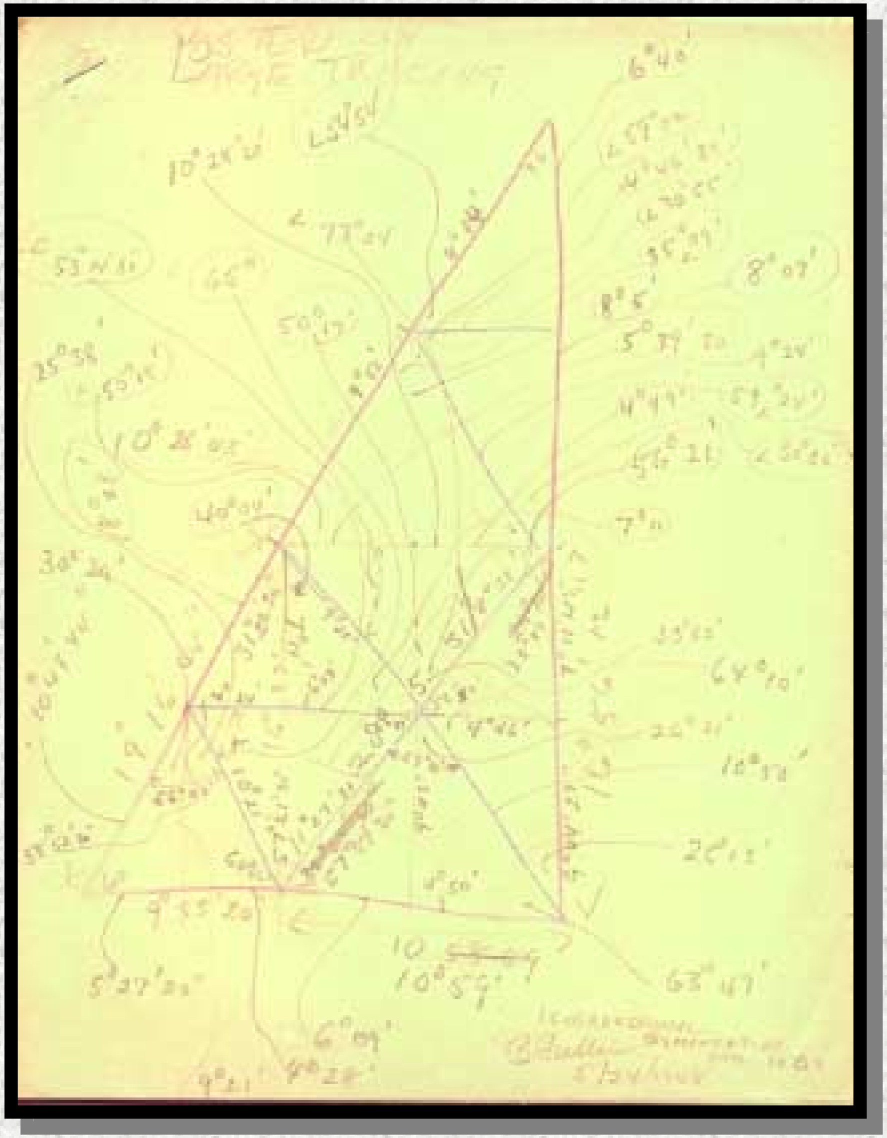
- **He further subdivided the basic unit of the 31 Great Circles into 4 right triangles.**

- **He used spherical trigonometry for his calculations.**

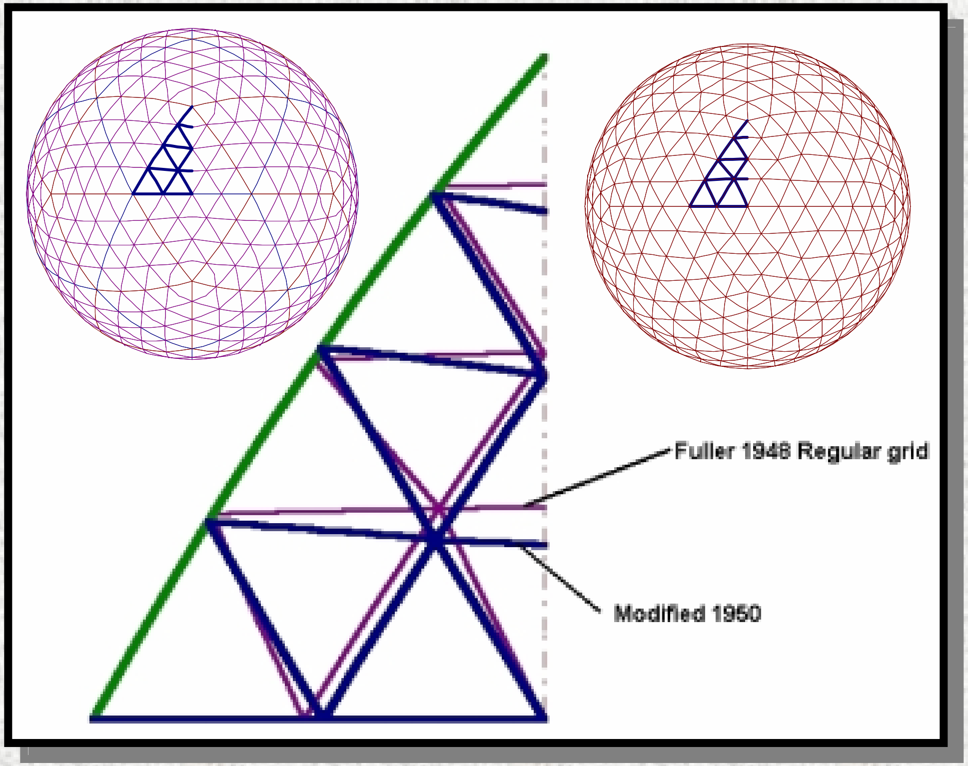
Fuller's Methods

- These triangles gave him the coordinates for constructing the three-way triangular grid of his geodesic polyhedra.





Fuller's Methods



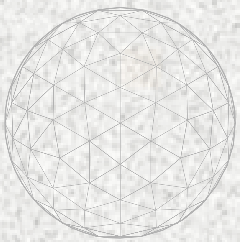
• The original three-way, shown on the right had an irregular pattern

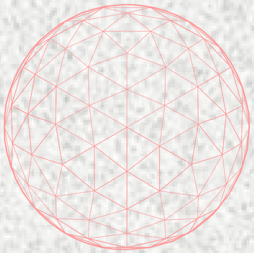
• It was modified and became known at the "Regular" triangulated grid.

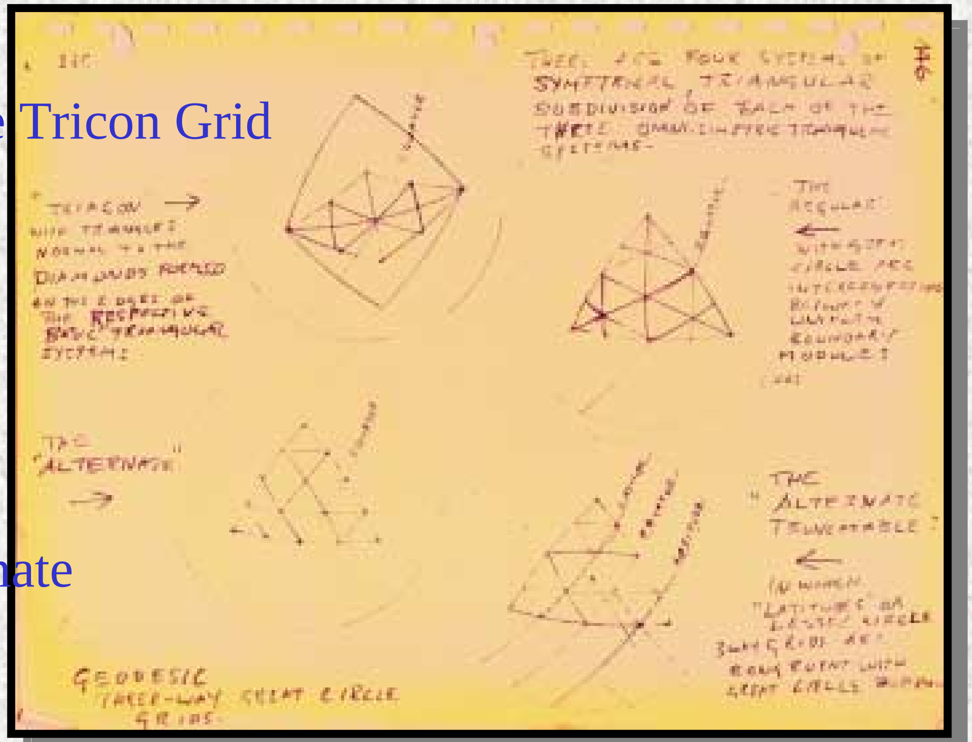
Fuller's Methods

- **Several other methods of generating three-way triangulated grids were developed and named:**

The Tricon Grid

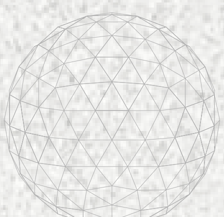


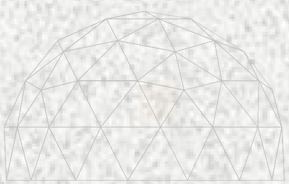




The Regular

The Alternate





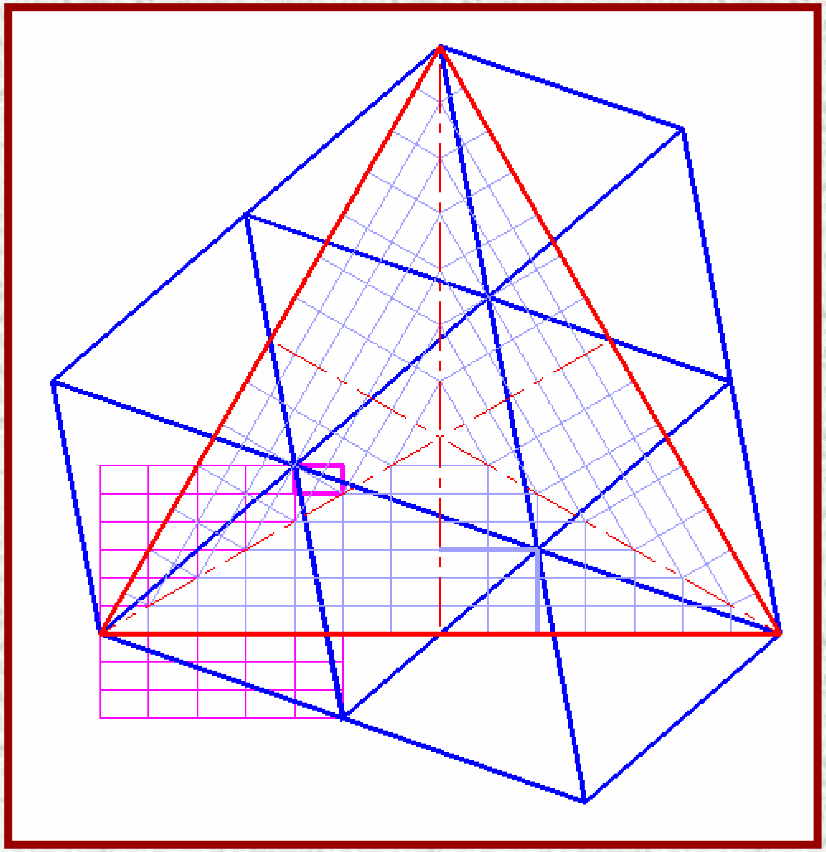
The Alternate Truncatable

Kitrick's Algorithms

- Source: Kitrick, Christopher J, "A Unified Approach to Class I, II, & III Geodesic Domes", IJSS, V 5, N 3&5, 1990 p 223-246

- Kitrick's mathematical approach will be described.

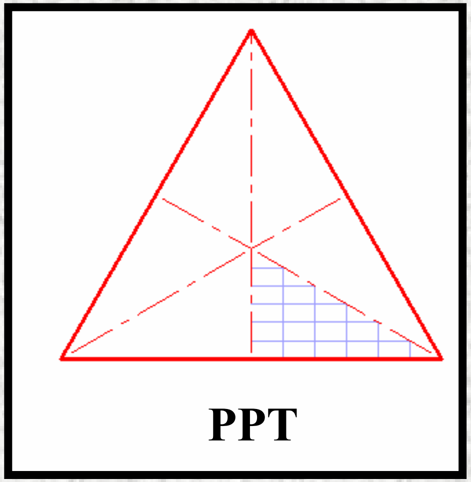
- It includes the gnomonic projection, Fuller's geometries and several other geodesic geometries



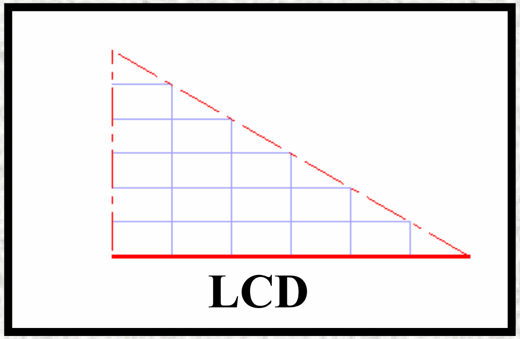
Kitrick's Algorithms

Methodology

• All concepts presented are applicable to the Schwarz triangles (LCD's) of the Icosahedron, Octahedron and Tetrahedron.



• It includes all classes and frequencies (b,c pairs).



• The basic approach involves the modular subdivision of the LCD triangle into a rectangular grid.

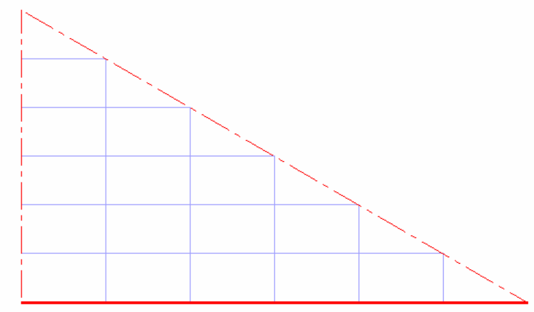
Kitrick's Algorithms

Methodology

- The approach uses a modular subdivision of the Schwarz triangle into a rectangular grid.

- The number of divisions along the PPT edge is referred to as the grid frequency (*f*)

- For every (*b*,*c*) pair there is a corresponding frequency.



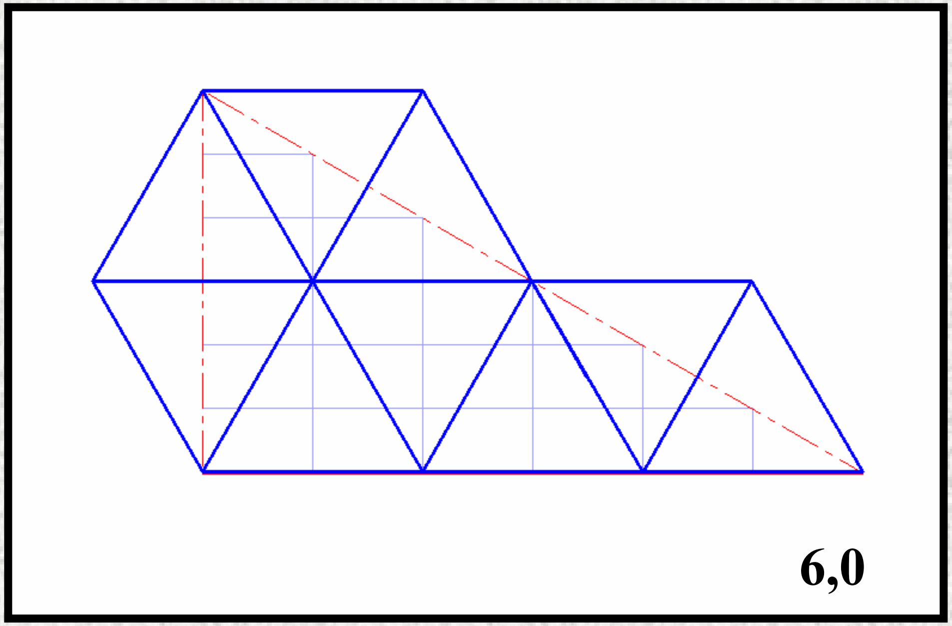
$f = 6$

Note: The grid frequency is not the same as the geodesic polyhedron frequency

Kitrick's Algorithms

Methodology

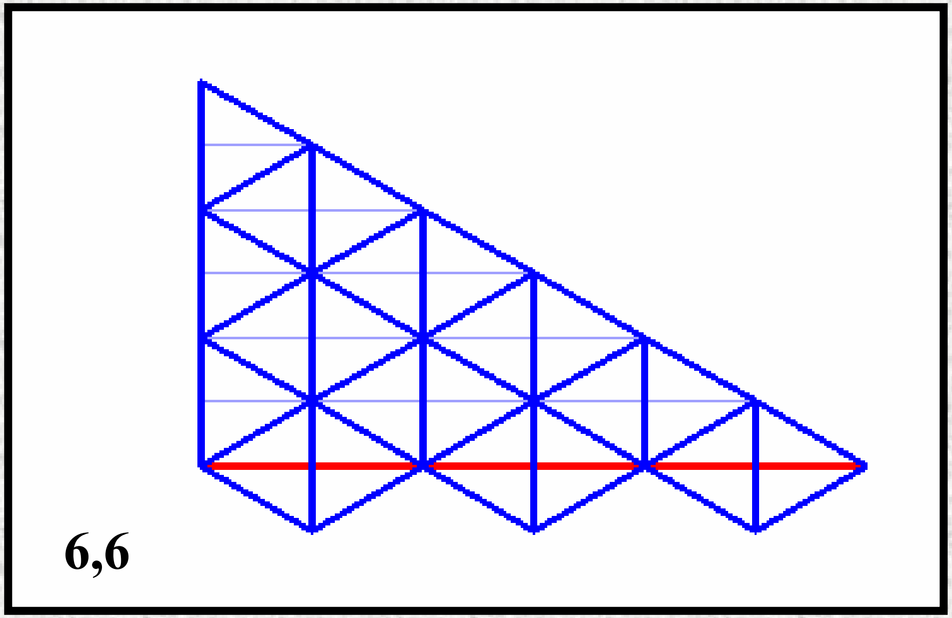
• For the Class I tessellation frequency equals *b/2* and each triangle is two cells wide and three high.



Kitrick's Algorithms

Methodology

• For the Class II tessellation frequency equals ***b*** and each triangle is one cells wide and two high.

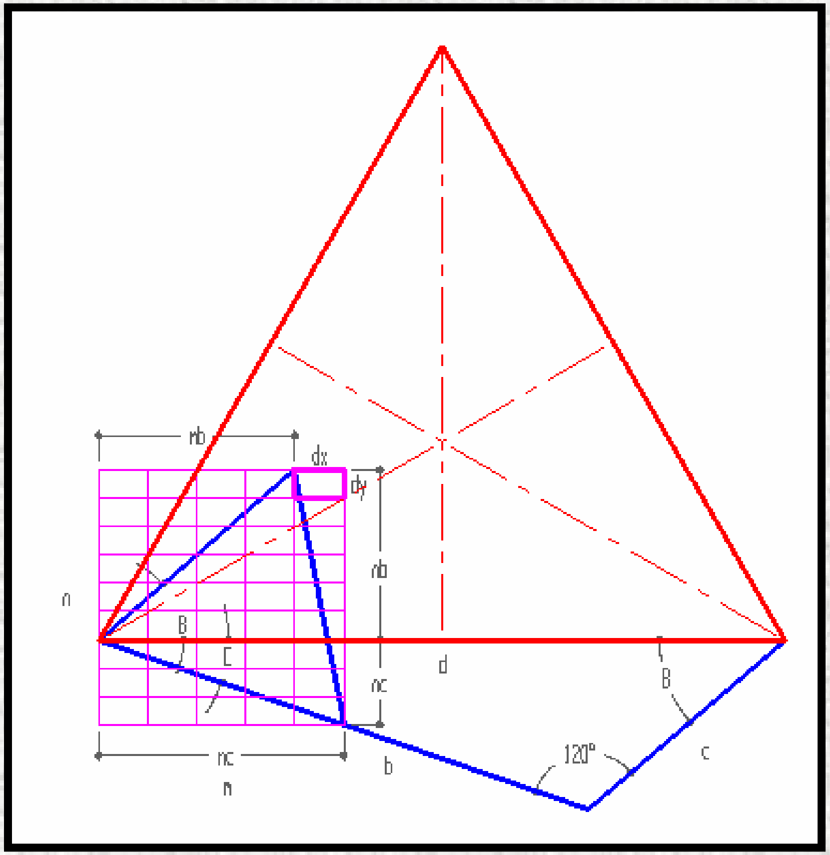


Kitrick's Algorithms

Methodology

- The Class III tessellation is more complex. It involves a skew angle to the grid.

- A set of offsets are applied to determine the correct frequency for the (b,c) pairs.

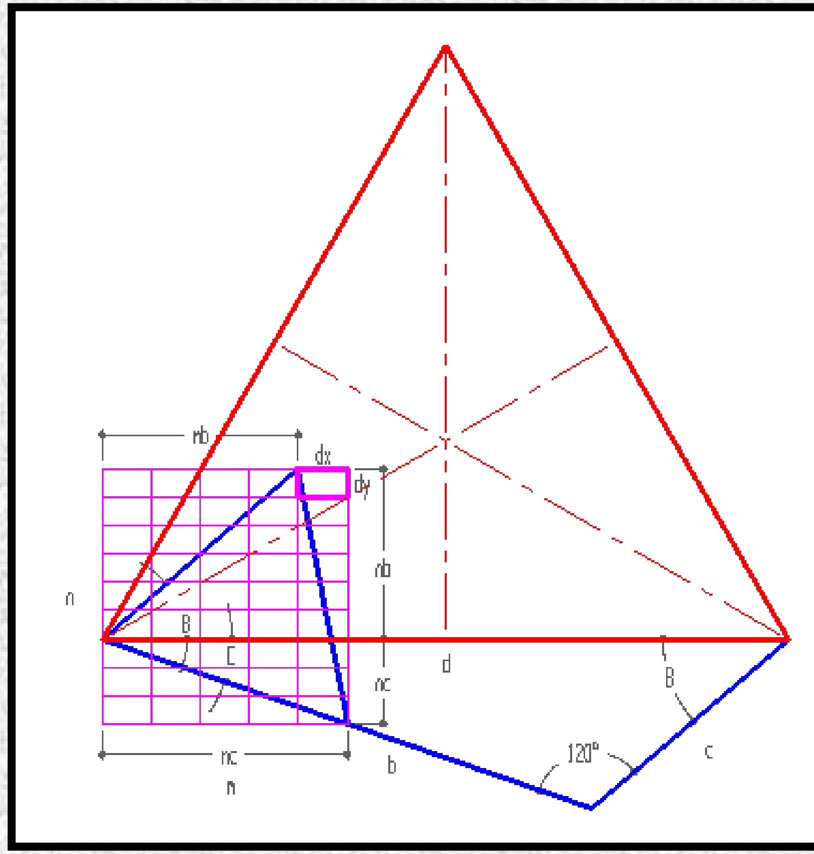


2,1

Kitrick's Algorithms

Methodology

- $d \quad = S(b^2 + c^2/4 + bc + \frac{3}{4}c^2)$
- $\cos B \quad = (b/2 + c)/d$
- $\cos C \quad = (c/2 + b)/d$
- $\sin B \quad = (S3b)/(2d)$
- $\sin C \quad = (S3c)/(2d)$
- $dx \quad = \cos C - \cos B$
- $dy \quad = S3/3dx$

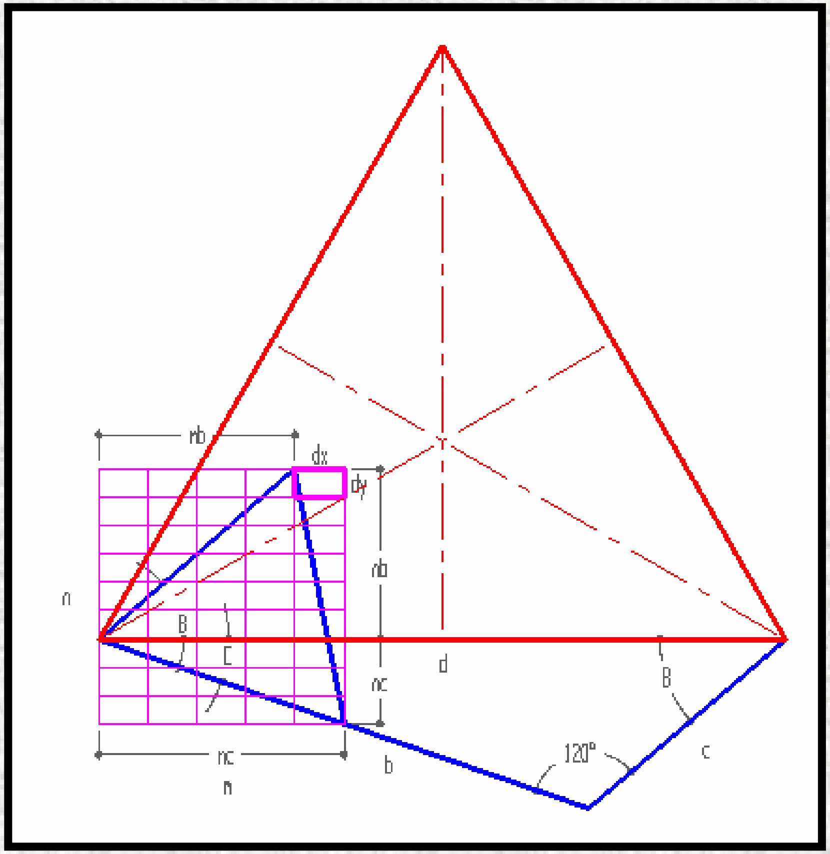


2,1

Kitrick's Algorithms

Methodology

- *m'* = *cosC(b-c)/dx*
  - *Note: b>c*
- *n'* = *(sinB + sinC)(b – c)/dy*
  - *Note: b>c*
- *m* = *m'/greatest common multiple (m', n')*
- *n* = *n'/greatest common multiple (m', n')*
- m = *cosC/m*
- j = S3/3 m

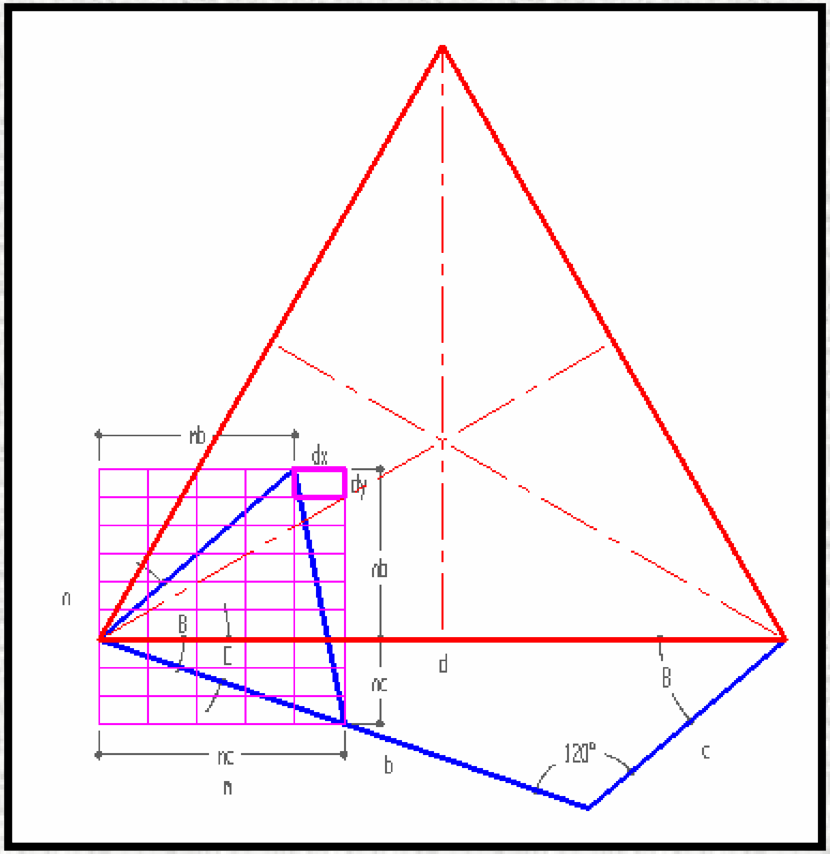


2,1

Kitrick's Algorithms

Methodology

- m_b = $\cos B/m$
- m_c = $\cos C/m$
- nb = $\sin B/j$
- n_c = $\sin C/j$
- f = $(bm_c + cm_b/2) = d(2m)$



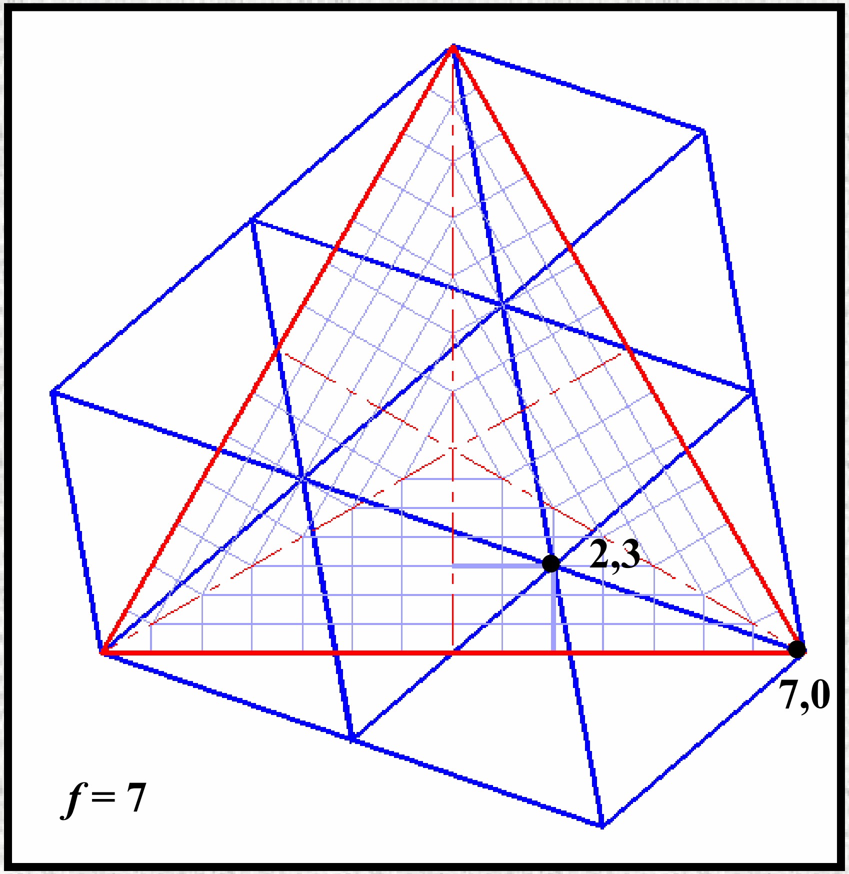
Kitrick's Algorithms

Methodology

- All grid intersections are defined by a coordinate pair given as **(i,j)** where:

  - $i + j < f$

- Any given geometrical method is a unique one-to-one mapping of **(i,j)** pairs to **(x,y,z)** coordinates.



$\{p,q+\}_{2,1}$
n= 3

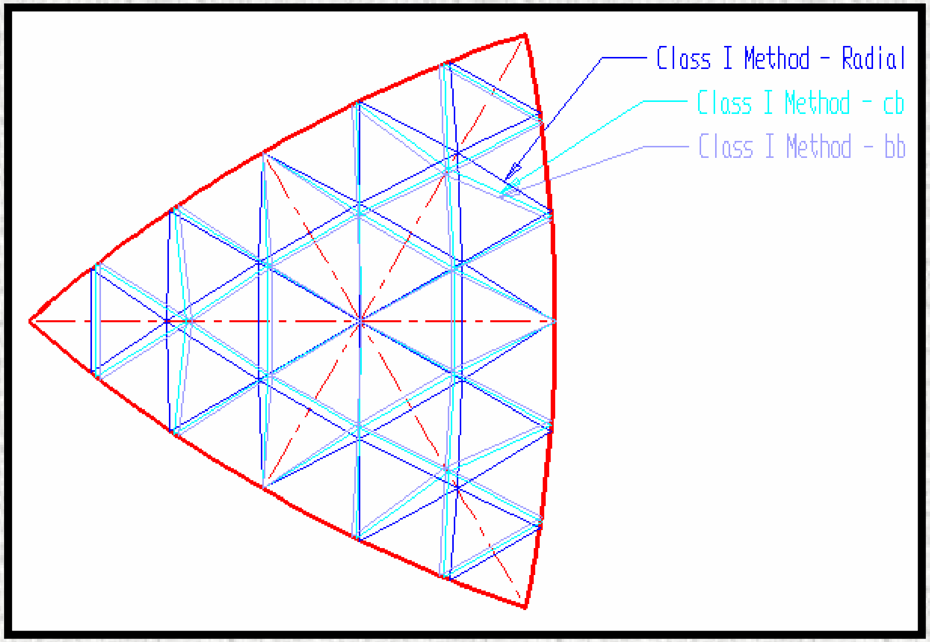
Kitrick‘s Algorithms

Mapping

- Many geometrical solutions have been found for triangulated spherical tessellations.

- Three solutions will be presented here:

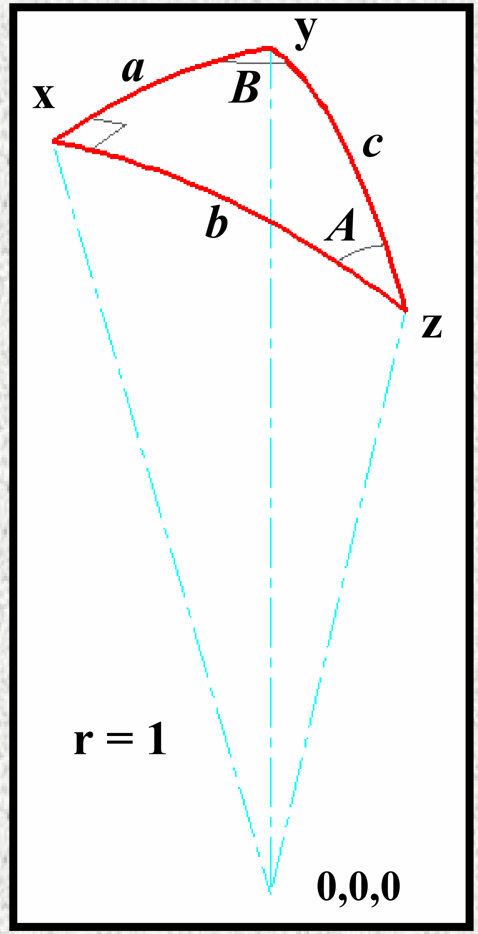
    - Method – radial
    - Method – cb
    - Method - bb



Kitrick's Algorithms

Mapping

- The spherical methods use the spherical form of the *LCD*'s using the following notations:
  - *a* - opposite arc side
  - *b* - adjacent arc side
  - *c* - hypotenuse arc side
  - *A* - angle opposite arc side *a*
  - *B* - angle opposite arc side *b*
  - (*i,j*) - integer coordinate of a point on the *LCD* grid
  - *f* - frequency of grid
  - $(x^0,y^0)_{axis}$ – angular equilivalent of (*i,j*): Subscript denotes side
  - (x,y,z) – 3d coordinates of (*i,j*) and $(x^0,y^0)_{axis}$



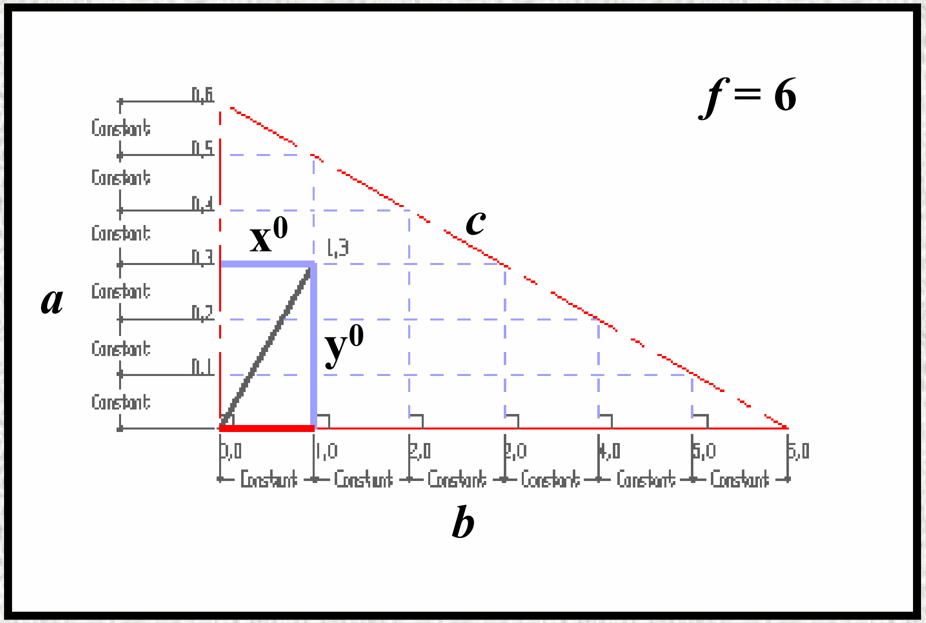
Kitrick's Algorithms

Mapping

- The radial method

- The ***LCD*** triangle is divided by ***f*** in its planer form and each intersection (***i,j***) is projected radially until it reaches the sphere surface

- Take the (***x',y',z'***) position on the plane and divide each of its components by the radius of the sphere to find the projected (***x,y,z***) coordinates on the sphere surface

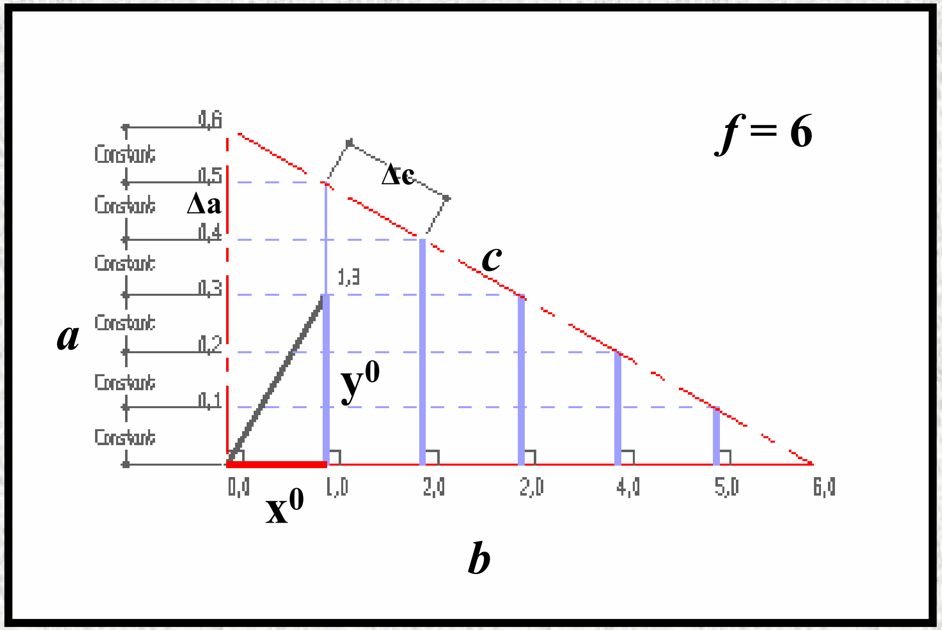


Kitrick's Algorithms

Mapping

- Method - cb
- Side c is subdivided by f into Δc arc segments
- From side c perpendicular arcs are dropped at each Δc interval to be perpendicular with side b
- All grid intersections lie on these arcs perpendicular to side b

$$
\begin{aligned}
\Delta c &= c/f \\
y^0 &= \arcsin(\sin(j\Delta c)\sin A) \\
x^0 &= b - \arctan(\tan(\Delta c(f-i))\cos A) \\
Axis &= x
\end{aligned}
$$



Kitrick's Algorithms

Mapping

- Method - bb

- Side ***b*** is subdivided by ***f*** into ***Δb*** arc segments

- At each ***Δb*** ***distance a perpendicular arc c is projected until it intersects side c***

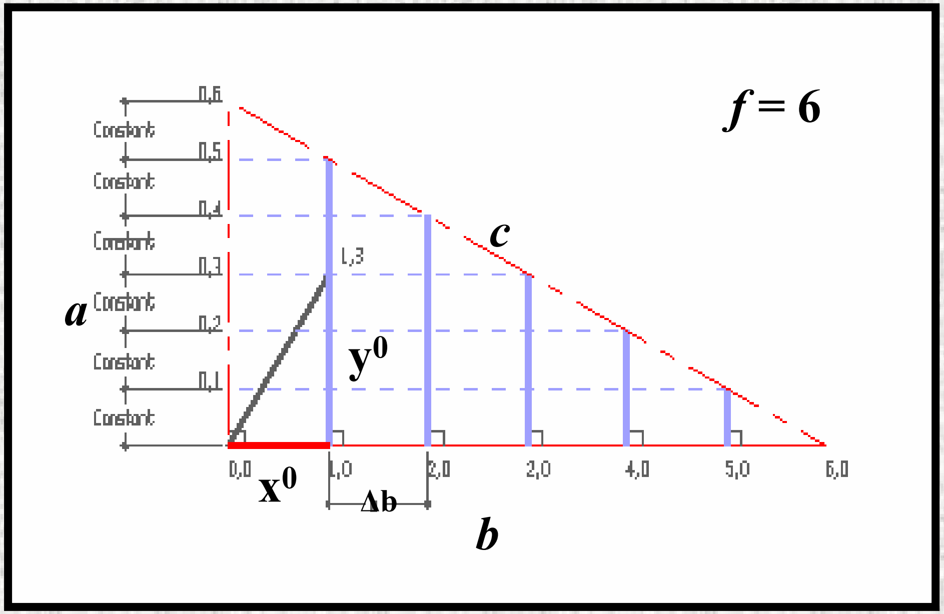
- All grid intersections lie on these arcs

$\Delta b = b/f$

$x^0 = i\Delta b$

$y^0 = \arctan(\sin(\Delta a)\tan A)$

$Axis = x$



Other Methods

• Kitrick included in his paper nine different geometrical methods for tessellating the sphere into a triangular grid.

• Bauersfeld, Fuller, Goldberg, Ginzburg, Stuart, Richter, Kirschenbaum, Clinton, Tarnai & Makai, Edmondson, Pavlov, Rebielak, Huybers, Trump, Fowler & Manolopolos, Shea, and others have contributed to enlarging the inventory of methods

• New methods are appearing at an accelerated rate coming from many divers fields.